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# Independent Set on graphs with maximum degree 3

Iyad A. Kanj\*

Fenghui Zhang†

## Abstract

Let  $G$  be an undirected graph with maximum degree at most 3 such that  $G$  does not contain either of the two graphs shown in Figure 1 as a subgraph. We prove that the independence number of  $G$  is at least  $n(G)/3 + nt(G)/63$ , where  $n(G)$  is the number of vertices in  $G$  and  $nt(G)$  is the number of nontriangle vertices in  $G$ . We show an application of the aforementioned combinatorial result to the area of parameterized complexity. We present a linear-time kernelization algorithm for the independent set problem on graphs with maximum degree at most 3 that computes a kernel of size at most  $630k/211 < 3k$ , where  $k$  is the lower bound on the size of the independent set sought.

## 1 Introduction

We consider the independent set problem on graphs of maximum degree at most 3, abbreviated IS-3: Given an undirected graph  $G$  with maximum degree at most 3 and a nonnegative integer  $k$ , decide if  $G$  has an independent set of cardinality at least  $k$ . The problem is known to be NP-complete [7].

We take a combinatorial approach to the problem, establishing lower bounds on the independence number (cardinality of a maximum independent set) of a graph of maximum degree at most 3 that excludes both of the two obstacle-graphs depicted in Figure 1 as subgraphs. Combinatorial results of a similar nature are very common in the literature. Brook’s theorem [3], published as early as 1941, implies that the independence number of a  $K_4$ -free graph  $G$  with maximum degree 3 is at least  $n(G)/3$ , where  $n(G)$  is the number of vertices in  $G$ . Staton showed in 1979 [11] that the independence number of a triangle-free graph  $G$  with maximum degree at most 3 is at least  $5n(G)/14$ . Staton’s lower bound for triangle-free graphs is tight, as shown by the example given in [6]. A simpler proof of Staton’s result was given by Jones in 1990 [10], and an even simpler proof was given by Heckman and Thomas in 2001 [9]. In their result [9], Heckman and Thomas define the notion of a *difficult component* in a graph, based on some “obstacle” subgraphs. They then prove that every triangle-free graph with maximum degree at most 3 has an independent number of at least  $(4n(G) - e(G) - \lambda(G))/7$ , where  $e(G)$  and  $\lambda(G)$  are the number of edges and the number of difficult components in  $G$ , respectively. They showed how their result implies Staton’s result [11].

Very recently (2008), Harant et al. [8] generalized Heckman and Thomas’ result to graphs of maximum degree at most 3 that may contain triangles. They define the notion of a *difficult block*, which is a block (a biconnected component) that is isomorphic to one of the four obstacle graphs given in Figure 3. They use the notion of difficult blocks to define the *bad components* of a graph, which are the components in which every block is either a difficult block or an edge between two

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difficult blocks. They then prove that the independence number of a  $K_4$ -free graph with maximum degree at most 3 is at least  $(4n(G) - e(G) - \lambda(G) - tr(G))/7$ , where  $\lambda(G)$  and  $tr(G)$  are the number of bad components and the number of vertex-disjoint triangles in  $G$ , respectively.

In the current paper we prove the following combinatorial result: if  $G$  is a graph with maximum degree at most 3 that does not contain either of the two obstacle graphs depicted in Figure 1 as a subgraph, then the independence number of  $G$  is at least  $n(G)/3 + nt(G)/63$ , where  $nt(G)$  is the number of nontriangle vertices in  $G$  (i.e., vertices that do not appear in any triangle). The technique employed in proving the aforementioned result is the following. We apply a sequence of operations to the graph  $G$  to obtain a graph  $G'$  with a much simpler structure. None of these operations decreases the number of nontriangle vertices in the graph, and each of these operations guarantees that the independence number of the graph to which the operation is applied is at least as large as that of the resulting graph plus one third the number of vertices removed by the operation. Finally, a lower bound of  $n(G')/3 + nt(G')/63$  is established on the independence number of  $G'$ , which implies a lower bound of  $n(G)/3 + nt(G)/63$  on the independence number of  $G$ .

The result derived in the current paper and that of Harant et al. [8] are somehow orthogonal, which makes it difficult to compare them. We note that it is possible that the result in the current paper yields a better lower bound on the independence number. For example, if the graph  $G$  contains more than  $n(G)/6$  vertex-disjoint triangles and is 3-regular (or almost 3-regular), then Harant et al.'s result [8] gives a lower bound of  $n(G)/3$  (or almost  $n(G)/3$ ) on the independence number. The graph  $G$  in this case can still contain many nontriangle vertices (up to  $n/2$  nontriangle vertices when  $tr(G) = n/6$ ), and hence the result in this paper implies a much better lower bound on the independence number in such cases.

The advantage of the combinatorial result in the current paper over the previous results is that the two obstacle structures depicted in Figure 1 can be pre-processed in polynomial time by any algorithm for IS-3, as shown in Section 3. This fact, in addition to some reduction rules that allow us to lower bound the value of  $nt(G)$ , yield a kernelization algorithm for the IS-3 problem that produces a kernel of size at most  $630k/211$  in  $O(k)$  time. We note that since a  $K_4$  subgraph must appear as a separate component in a graph of maximum degree 3, Brook's theorem [3] implies a kernel of size at most  $3k$  for IS-3. Improving on the  $3k$  upper bound on the kernel size has been an open problem. The  $630k/211$  upper bound on the kernel size for IS-3 implies an upper bound of  $630k/419$  on the kernel size for the vertex cover problem on graphs with maximum degree at most 3, abbreviated VC-3, as shown in Section 7. Both results are in line with the recent progress in deriving lower and upper bounds on the kernel size for certain fixed-parameter tractable problems [1, 2].

The paper is organized as follows. In Section 2 we give the necessary notations and terminologies used throughout the paper. In Section 3 we give a set of reduction rules that will constitute the core of the kernelization algorithm for IS-3, given in Section 6, and that will also be part of the graph operations given in Section 4. The graph operations given in Section 4 will be used to prove the main combinatorial result in Section 5. The kernelization algorithm is given in Section 6, and the lower bound result on the kernel size for VC-3 is given in Section 7.

## 2 Preliminaries

We assume familiarity with the basic notations and terminologies about graphs. For more information, we refer the reader to West [12]. We only consider simple undirected graphs in this paper.

For a graph  $G$  we denote by  $V(G)$  and  $E(G)$  the set of vertices and edges of  $G$ , respectively;  $n(G) = |V(G)|$  and  $e(G) = |E(G)|$  are the number of vertices and edges in  $G$ . A set of vertices in  $V(G)$  is said to be an *independent set* if no edge in  $E(G)$  exists between any two vertices in this set. By  $\alpha(G)$  we denote the independence number of  $G$ ; that is, the size of a maximum independent set in  $G$ .

For a set of vertices  $S \subseteq V(G)$ , we denote by  $G[S]$  the subgraph of  $G$  induced by the set of vertices in  $S$ . For a vertex  $v \in G$ ,  $G - v$  denotes  $G[V(G) \setminus \{v\}]$ , and for a subset of vertices  $S \subseteq V(G)$ ,  $G - S$  denotes  $G[V(G) \setminus S]$ . For two vertices  $u, v \in V(G)$ , we denote by  $G - (u, v)$  the graph  $(V(G), E(G) \setminus \{(u, v)\})$ , and by  $G + (u, v)$  the *simple* graph  $(V(G), E(G) \cup \{(u, v)\})$ .

The *degree* of a vertex  $v$  in  $G$ , denoted  $d(v)$ , is the number of edges in  $G$  that are incident to  $v$ . The *degree* of  $G$ , denoted  $\Delta(G)$ , is defined as  $\Delta(G) = \max\{d(v) \mid v \in G\}$ .

Call a vertex  $v \in G$  a *triangle vertex* if  $v$  is a vertex of some triangle in  $G$ ; otherwise call  $v$  a *nontriangle vertex*. We denote the number of vertex-disjoint triangles in  $G$  by  $tr(G)$ , and the number of nontriangle vertices in  $G$  by  $nt(G)$ .

Two triangles in a graph are said to *share an edge* if the two triangles have exactly two vertices in common. Two triangles are said to be *adjacent* if the two triangles do not have any common vertex and a vertex in one of the triangles is adjacent to a vertex in the other triangle. Note that if a graph has maximum degree at most 3, then no two triangles in the graph can have exactly one vertex in common.

A *parameterized problem* is a set of instances of the form  $(x, k)$ , where  $x \in \Sigma^*$  for a finite alphabet set  $\Sigma$ , and  $k$  is a non-negative integer called the *parameter* [5]. A parameterized problem  $Q$  is *kernelizable* [5] if there exists a polynomial-time computable reduction that maps an instance  $(x, k)$  of  $Q$  to another instance  $(x', k')$  of  $Q$  such that: (1)  $|x'| \leq g(k)$  for some recursive function  $g$ , (2)  $k' \leq k$ , and (3)  $(x, k)$  is a yes-instance of  $Q$  if and only if  $(x', k')$  is a yes-instance of  $Q$ . The instance  $x'$  is called the *kernel* of  $x$ . For more information on parameterized complexity and kernelization we refer the reader to [5].

The INDEPENDENT SET problem on graphs of maximum degree at most 3, abbreviated IS-3, is defined as follows:

IS-3. Given an undirected graph  $G$  with  $\Delta(G) \leq 3$ , and a nonnegative integer  $k$ , determine if  $G$  has an independent set of size at least  $k$ .

### 3 Reductions

Let  $(G, k)$  be an instance of IS-3.

**Fact 3.1** *Let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = 2$ . Then there exists a maximum independent set of  $G$  that contains  $u$ .*

The validity of the following reduction rule is implied by Fact 3.1.

**Reduction Rule 3.1** *Let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = 2$ . Then include  $u$  in the maximum independent set for  $G$  and set  $G := G - \{u, v, w\}$  and  $k := k - 1$ .*

**Fact 3.2** *Let  $(u, v, w)$  and  $(p, v, w)$  be two triangles in  $G$  that share an edge  $(v, w)$ . Then there exists a maximum independent set of  $G$  that excludes  $v$  (or  $w$ ).*

The validity of the following reduction rule is implied by Fact 3.2:

**Reduction Rule 3.2** Let  $(u, v, w)$  and  $(p, v, w)$  be two triangles in  $G$  that share an edge  $(v, w)$ . Then set  $G := G - v$  (i.e., vertex  $v$  can be removed from  $G$ ).

We assume for the remaining discussion in this section that no triangle in  $G$  contains a vertex of degree 2, and that no two triangles in  $G$  share an edge. Therefore, since no two (distinct) triangles in  $G$  can share exactly one vertex, any two triangles in  $G$  are vertex-disjoint.

A sequence of distinct triangles  $T_1, \dots, T_\ell$ ,  $\ell \geq 1$ , in  $G$  is said to form a *path of triangles* if either  $\ell = 1$ , or if  $\ell > 1$  and triangle  $T_i$  is adjacent to  $T_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . A path of triangles  $T_1, \dots, T_\ell$  is said to be a *cycle of triangles* if either  $\ell > 2$  and  $T_1$  and  $T_\ell$  are adjacent, or  $\ell = 2$  and (some) two vertices of  $T_1$  are neighbors of two vertices of  $T_2$  (i.e., there are at least two edges between the vertices of  $T_1$  and the vertices of  $T_2$ ). The *length* of a path/cycle of triangles is the number of triangles in it. A path of triangles is *maximal* if it is maximal under containment.

**Lemma 3.3** Let  $T_1, \dots, T_\ell$  be a cycle of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ ,  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ , and  $u_\ell$  is adjacent to  $v_1$ . Then there exists a maximum independent set of  $G$  that contains  $\{v_1, \dots, v_\ell\}$ .

PROOF. Observe first that all the neighbors of the vertices  $\{v_1, \dots, v_\ell\}$  are vertices from triangles  $T_1, \dots, T_\ell$ . If  $I_{max}$  is a maximum independent set of  $G$ , then  $I_{max}$  contains at most one vertex from each of triangles  $T_1, \dots, T_\ell$ , and hence  $I_{max}$  contains at most  $\ell$  vertices from triangles  $T_1, \dots, T_\ell$ . It follows from the previous statements that if we replace the vertices in  $I_{max} \cap (\bigcup_{i=1}^\ell V(T_i))$  with  $\{v_1, \dots, v_\ell\}$ , we obtain a maximum independent set of  $G$  that contains  $\{v_1, \dots, v_\ell\}$ .  $\square$

The validity of the following reduction rule is implied by Lemma 3.3:

**Reduction Rule 3.3** Let  $T_1, \dots, T_\ell$  be a cycle of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ ,  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ , and  $u_\ell$  is adjacent to  $v_1$ . Then include vertices  $\{v_1, \dots, v_\ell\}$  in the maximum independent set, and set  $G := G - \bigcup_{i=1}^\ell V(T_i)$  and  $k := k - \ell$ .

**Lemma 3.4** Let  $T_1, \dots, T_\ell$  with  $\ell > 1$  be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ . Suppose that  $w_1$  and  $w_\ell$  share a common neighbor  $x$ ,  $v_1$  and  $u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  share a common neighbor  $z$ . Then there exists a maximum independent set of  $G$  that contains the vertices  $x, y$ , and the set of vertices  $\{v_2, \dots, v_\ell\}$ .

PROOF. Let  $I_{max}$  be a maximum independent set of  $G$ .

If  $I_{max}$  contains  $z$ , then  $I_{max}$  excludes both  $x$  and  $y$ , and by maximality,  $I_{max}$  contains exactly one vertex from each of triangles  $T_1, \dots, T_\ell$ . It is easy to see that we can replace  $z$  and the  $k$  vertices from  $T_1, \dots, T_\ell$  in  $I_{max}$ , with  $\{x, y\} \cup \{v_2, \dots, v_\ell\}$  to obtain an independent set of  $G$  of the same cardinality as  $I_{max}$ , and hence this independent set is a maximum independent set of  $G$ .

If  $I_{max}$  excludes  $z$ , then since the maximum independent set of the subgraph of  $G$  induced by the set of vertices  $\bigcup_{i=1}^\ell V(T_i) \cup \{x, y\}$  has size  $\ell + 1$ ,  $I_{max}$  contains exactly  $\ell + 1$  vertices from  $\bigcup_{i=1}^\ell V(T_i) \cup \{x, y\}$ . Those  $\ell + 1$  vertices can be replaced with the vertices in  $\{x, y\} \cup \{v_2, \dots, v_\ell\}$ , to obtain a maximum independent set of  $G$  containing vertices  $x, y$ , and the set of vertices  $\{v_2, \dots, v_\ell\}$ .

It follows that there exists a maximum independent set of  $G$  that contains vertices  $x, y$ , and the set of vertices  $\{v_2, \dots, v_\ell\}$ . This completes the proof.  $\square$

The validity of the following reduction rule is implied by Lemma 3.4:

**Reduction Rule 3.4** Let  $T_1, \dots, T_\ell$  with  $\ell > 1$  be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ . If  $w_1$  and  $w_\ell$  share a common neighbor  $x$ ,  $v_1$  and  $u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  share a common neighbor  $z$ , then include vertices  $x$ ,  $y$ , and  $v_2, \dots, v_\ell$ , in the maximum independent set and set  $G := G - (\bigcup_{i=1}^\ell V(T_i) \cup \{x, y, z\})$  and  $k := k - \ell - 1$ .

**Definition 3.1** Call a graph *reduced* if none of Reduction Rules 3.1–3.4 applies to the graph.

Let  $G$  be a reduced graph. A *tree of triangles* in  $G$  is a set of triangles such that the subgraph of  $G$  induced by the vertices of the triangles in this set is connected. A tree of triangles is *maximal* if it is maximal under set containment. We have the following lemma:

**Lemma 3.5** Let  $G$  be a reduced graph, and let  $\mathcal{T}$  be nonempty maximal tree of triangles in  $G$ . Then the number of edges whose one endpoint is vertex in a triangle in  $\mathcal{T}$  and whose other endpoint is a nontriangle vertex in  $G$  is at least  $|\mathcal{T}| + 2$ .

PROOF. The statement of the theorem follows by a standard inductive proof on the number of triangles in  $\mathcal{T}$ .  $\square$

**Lemma 3.6** Let  $G$  be a reduced graph. Then the number of nontriangle vertices  $nt(G)$  satisfies  $nt(G) \geq n(G)/10$ .

PROOF. By Lemma 3.5, there are at least  $(|\mathcal{T}| + 2)$  edges between any maximal tree  $\mathcal{T}$  and nontriangle vertices in  $G$ . The statement now follows by summing over all maximal trees in  $G$ , and noting that the number of vertices in any maximal tree  $\mathcal{T}$  is  $3|\mathcal{T}|$  (because  $G$  is reduced, and hence no two triangles share vertices/edges), and that every nontriangle vertex has degree at most 3.  $\square$

## 4 Operations

Let  $G$  be a graph with  $\Delta(G) \leq 3$ , and such that  $G$  does not contain any of the two graphs depicted in Figure 1 as an induced subgraph. These two graphs present an obstacle for the combinatorial lower bound that we derive on the independence number of  $G$ . We call the two graphs depicted in Figure 1 the *obstacle graphs*. The graph on the left of Figure 1 is referred to as the *big obstacle*, and that on the right of Figure 1 as the *small obstacle*. Note that the degree of vertex  $z$  in  $G$  (i.e.,  $d(z)$ ) in the big obstacle could be 2 or 3. Similarly, the degree of each of the two vertices  $p$  and  $u$  in the small obstacle could be 2 or 3. We will say that  $G$  is *obstacle-free* to mean that  $G$  does not contain a small obstacle nor a big obstacle as a subgraph. Note that since the degree of  $G$  is at most 3,  $G$  contains the big obstacle as an induced subgraph if and only if it contains it as a subgraph. Note also that since  $G$  does not contain a small obstacle,  $G$  is  $K_4$ -free.

In what follows we introduce a set of graph operations to be applied to the graph  $G$  to obtain a “simplified” graph. We will then derive in the next section a lower bound on the independence number of the simplified graph, and use that to derive a lower bound on the independence number of  $G$ . To do so, we need to keep track of how each operation affects the number of vertices, the number of non triangle vertices, and the independence number of the graph  $G$ . For convenience, if an operation, or a set of operations, is applied to  $G$  to obtain a graph  $G'$ , we will denote by  $\delta_n(G)$ ,

$\delta_{nt(G)}$ , and  $\delta_{\alpha(G)}$  the values  $n(G) - n(G')$ ,  $nt(G) - nt(G')$ , and  $\alpha(G) - \alpha(G')$ , respectively. It is also essential that none of these operations when applied to a graph that is obstacle-free produces an obstacle in the resulting graph, or produces a graph with maximum degree larger than 3. The following observation will be useful in proving the previous statements:

**Observation 4.1** *Let  $G$  be a graph with  $\Delta(G) \leq 3$  such that  $G$  is obstacle-free. Then for any subset of vertices  $S \subseteq V(G)$ , the subgraph  $G - S$  of  $G$  has maximum degree at most 3 and is obstacle-free.*

PROOF. It is clear that  $G - S$  has maximum degree at most 3. The fact that  $G - S$  is obstacle-free follows from the fact that  $\Delta(G) \leq 3$ , and that every vertex in an obstacle, except  $z, p, u$  which could have degree 2 or 3, has degree 3. Therefore, if an obstacle graph did not already exist in  $G$  then the removal of a subset of vertices from  $G$  will not create an obstacle graph.  $\square$

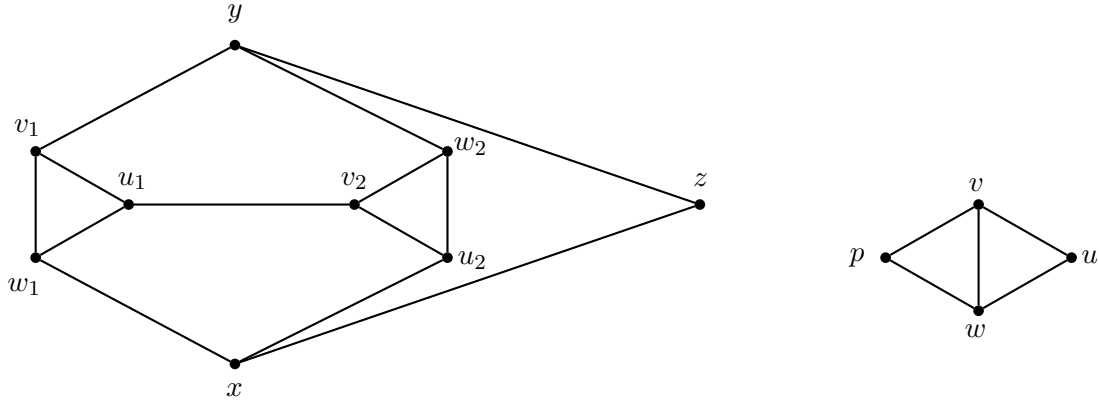


Figure 1: The obstacle graphs. The graph on the left is referred to as the *big obstacle* and that on the right as the *small obstacle*. The degree of vertices  $z, p, u$  in  $G$  could be either 2 or 3.

Each of the operations that follow is justified by the lemma preceding it. The proofs of these lemmas are delegated to the appendix for lack of space.

**Lemma 4.2** *Let  $(u, v, w)$  be a triangle in  $G$  such that one of its vertices is of degree 2. Let  $G' = G - \{u, v, w\}$ . Then  $\delta_n(G) = 3$ ,  $\delta_{\alpha(G)} = 1$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.*

PROOF. Since  $G'$  is obtained from  $G$  by removing 3 vertices, we have  $\delta_n(G) = 3$ . The fact that  $\delta_{\alpha(G)} = 1$  follows from Fact 3.1, and since  $u, v, w$  are triangle vertices, we have  $\delta_{nt(G)} \leq 0$ . Finally, by Observation 4.1,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.  $\square$

**Operation 4.1** *Let  $(u, v, w)$  be a triangle in  $G$  such that one of its vertices is of degree 2. Then set  $G := G - \{u, v, w\}$ .*

**Lemma 4.3** *Let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = d(v) = d(w) = 3$ . Let  $u', v'$ , and  $w'$  be the neighbors of  $u, v, w$ , respectively that are not in the set  $\{u, v, w\}$ . Suppose that two vertices in  $\{u', v', w'\}$  are adjacent, and let  $G' = G - \{u, v, w\}$ . Then  $\delta_n(G) = 3$ ,  $\delta_{\alpha(G)} \geq 1$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.*

PROOF. It is clear that  $\delta_{n(G)} = 3$ . Since two vertices in  $\{u', v', w'\}$  are adjacent, say vertices  $u'$  and  $v'$ , any maximum independent set  $I'$  of  $G'$  contains at most one vertex from  $\{u', v'\}$ . If  $u' \notin I'$ , then  $I' \cup \{u\}$  is an independent set of  $G$ . On the other hand, if  $v' \notin I'$ , then  $I' \cup \{v\}$  is an independent set of  $G$ . It follows that  $\delta_{\alpha(G)} \geq 1$ . Since the removed vertices  $u, v, w$  are all triangle vertices, we have  $\delta_{nt(G)} \leq 0$ . By Observation 4.1,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.  $\square$

**Operation 4.2** Let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = d(v) = d(w) = 3$ . Let  $u', v',$  and  $w'$  be the neighbors of  $u, v, w$ , respectively that are not in the set  $\{u, v, w\}$ . If two vertices in  $\{u', v', w'\}$  are adjacent, then set  $G := G - \{u, v, w\}$ .

**Lemma 4.4** Let  $T_1, \dots, T_\ell$  be a cycle of triangles, where  $T_i = (u_i, v_i, w_i)$ ,  $i = 1, \dots, \ell$ ,  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ , and  $u_\ell$  is adjacent to  $v_1$ . Let  $G'$  be the subgraph obtained from  $G$  by removing the vertices in  $\bigcup_{i=1}^\ell V(T_i)$  (i.e.,  $G' = G - \bigcup_{i=1}^\ell V(T_i)$ ). Then  $\delta_{n(G)} = 3\ell$ ,  $\delta_{\alpha(G)} \geq \ell$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.

PROOF. Since  $G'$  is obtained from  $G$  by removing exactly  $3\ell$  vertices, we have  $\delta_{n(G)} = 3\ell$ . The fact that  $\delta_{\alpha(G)} \geq \ell$  follows from Lemma 3.3. Since none of the vertices removed is a nontriangle vertex, we have  $\delta_{nt(G)} \leq 0$ . The facts that  $\Delta(G') \leq 3$  and that  $G'$  is obstacle-free follow from Observation 4.1.  $\square$

**Operation 4.3** Let  $T_1, \dots, T_\ell$  be a cycle of triangles, where  $T_i = (u_i, v_i, w_i)$ ,  $i = 1, \dots, \ell$ ,  $u_i$  is adjacent to  $v_{i+1}$  for  $i = 1, \dots, \ell - 1$ , and  $u_\ell$  is adjacent to  $v_1$ . Then set  $G := G - \bigcup_{i=1}^\ell V(T_i)$ .

**Lemma 4.5** Let  $T_1, \dots, T_\ell$ ,  $\ell > 2$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . Suppose that  $w_1$  and  $w_\ell$  share a common neighbor  $x$ ,  $v_1$  and  $u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  share a common neighbor  $z$ . Let  $G'$  be the graph resulting from  $G$  after removing the set of vertices  $V(T_1) \cup \bigcup_{i=3}^\ell V(T_i)$  and adding the two edges  $(x, v_2)$  and  $(y, u_2)$ ; that is,  $G' = (G - V(T_1) - \bigcup_{i=2}^\ell V(T_i)) + (x, v_2) + (y, u_2)$ . Then  $\delta_{n(G)} = 3(\ell - 1)$ ,  $\delta_{\alpha(G)} \geq \ell - 1$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.

PROOF. The fact that  $\delta_{n(G)} = 3(\ell - 1)$  follows from the fact that  $|V(G)| = |V(G')| + |V(T_1)| + |\bigcup_{i=3}^\ell V(T_i)|$ , and that  $|V(T_1)| + |\bigcup_{i=3}^\ell V(T_i)| = 3(\ell - 1)$ .

Now to show that  $\delta_{\alpha(G)} \geq \ell - 1$ , let  $I'$  be a maximum independent set of  $G'$ . If  $I'$  contains both  $x$  and  $y$ , then  $I'$  must exclude  $v_2$  and  $u_2$ . In this case  $I = I' \cup \{u_1\} \cup \{v_3, \dots, v_\ell\}$  is an independent set in  $G$  of size  $|I'| + (\ell - 1)$ . If  $I'$  excludes  $x$ , then  $I = I' \cup \{w_1, w_\ell\} \cup \{u_3, \dots, u_{\ell-1}\}$  if  $\ell > 3$  and  $I = I' \cup \{w_1, w_3\}$  if  $\ell = 3$ , is an independent set in  $G$  of size  $|I'| + (\ell - 1)$ . If  $I'$  excludes  $y$ , then  $I = I' \cup \{v_1\} \cup \{u_3, \dots, u_\ell\}$  is an independent set in  $G$  of size  $|I'| + (\ell - 1)$ .

It follows that  $G$  has an independent set of size  $\alpha(G') + \ell - 1$ , and hence,  $\delta_{\alpha(G)} \geq \ell - 1$ .

Due to the addition of edges  $(x, v_1)$  and  $(y, u_2)$ , the only vertices in  $G'$  whose degrees could have increased are vertices  $x, v_2, y, u_2$ . However, at least one neighbor of each of these vertices was removed from  $G$ ; therefore, the degree of each of those vertices is at most 3 in  $G'$ . It follows that  $\Delta(G') \leq 3$ . Now since all vertices removed from  $G$  are triangle vertices, to show that  $\delta_{nt(G)} \leq 0$ , it suffices to show that the addition of the two edges  $(x, v_2)$  and  $(y, u_2)$  does not create any triangles. The addition of these edges can create a triangle only if  $x$  or  $y$  is a neighbor in  $G$  of a vertex of  $V(T_2)$ . The neighbors of  $x$  in  $G$  are  $w_1, w_\ell, z$  and those of  $y$  are  $v_1, u_\ell, z$ . Since the triangles are



all vertex-disjoint, and since  $z$  is a nontriangle vertex ( $z$  is adjacent to both  $x$  and  $y$ ), neither  $x$  nor  $y$  can be a neighbor in  $G$  of a vertex in  $V(T_2)$ . To show that  $G'$  is obstacle-free, similar to the above, it suffices to show that the addition of edges  $(x, v_2)$  and  $(y, u_2)$  does not create obstacles. It is easy to see that the addition of these edge cannot create a small obstacle because each of these two edges has one endpoint that is a nontriangle vertex in  $G'$  ( $x$  and  $y$ ). Now each of  $x$  and  $y$  is a degree-2 vertex in  $G'$ , and is adjacent to exactly one triangle vertex in  $G'$  (since  $z$  is a nontriangle vertex in  $G'$ ). Therefore, neither  $x$  nor  $y$  can be a vertex of a big obstacle in  $G'$ , and hence the addition of the two edges  $(x, v_1)$  and  $(y, u_2)$  does not create obstacle graphs in  $G'$ .  $\square$

**Operation 4.4** Let  $T_1, \dots, T_\ell$ ,  $\ell > 2$  be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . If  $w_1$  and  $w_\ell$  share a common neighbor  $x$ ,  $v_1$  and  $u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  share a common neighbor  $z$ , then set  $G := (G - (V(T_1) \cup \bigcup_{i=3}^\ell V(T_i))) + (x, v_2) + (y, u_2)$ .

**Lemma 4.6** Let  $T_1, \dots, T_\ell$ ,  $\ell > 1$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . Suppose that  $w_1$  and  $w_\ell$  share a common neighbor  $x$  and  $v_1$  and  $u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  do not share a neighbor. Let  $G'$  be the graph resulting from  $G$  by removing the set of vertices  $\bigcup_{i=1}^\ell V(T_i)$  and adding the edge  $(x, y)$  (if  $(x, y)$  is not already an edge); that is  $G' = (G - \bigcup_{i=1}^\ell V(T_i)) + (x, y)$ . Then  $\delta_{n(G)} = 3\ell$ ,  $\delta_{\alpha(G)} \geq \ell$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.

PROOF. The fact that  $\delta_{n(G)} = 3\ell$  follows from the fact that  $|V(G)| = |V(G')| + |\bigcup_{i=1}^\ell V(T_i)|$ , and that  $\bigcup_{i=1}^\ell V(T_i)$  contains precisely  $3\ell$  vertices. To show that  $\delta_{\alpha(G)} \geq \ell$ , let  $I'$  be a maximum independent set of  $G'$ . Since  $(x, y) \in E(G')$ ,  $I'$  contains at most one vertex from  $\{x, y\}$ . If  $I'$  excludes  $x$ , then  $I' \cup \{w_1\} \cup \{v_i : i = 2, \dots, \ell\}$  is an independent set in  $G$  of size  $|I'| + \ell = \alpha(G') + \ell$ . On the other hand, if  $I'$  excludes  $y$ , then  $I' \cup \{u_i : i = 1, \dots, \ell\}$  is an independent set in  $G$  of size  $|I'| + \ell = \alpha(G') + \ell$ . It follows that  $G$  has an independent set of size  $\alpha(G') + \ell$ , and hence,  $\delta_{\alpha(G)} \geq \ell$ .

Since vertices  $x$  and  $y$  do not share a neighbor in  $G$ , the addition of edge  $(x, y)$  will not create a triangle. This, together with the fact that all vertices removed are triangle vertices, imply that  $\delta_{nt(G)} \leq 0$ . Now since both  $x$  and  $y$  are nontriangle vertices of degree at most 2 in  $G'$  (since two neighbors of each of  $x, y$  were removed), edge  $(x, y)$  cannot be an edge in an obstacle graph in  $G'$ , and hence its addition does not create obstacle graphs. Moreover, due to the fact that each of  $x$  and  $y$  has degree at most 2 in  $G'$ , we have  $\Delta(G') \leq 3$ .  $\square$

**Operation 4.5** Let  $T_1, \dots, T_\ell$ ,  $\ell > 1$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . Suppose that  $w_1$  and  $w_\ell$  share a common neighbor  $x$  and  $v_1, u_\ell$  share a common neighbor  $y$ , and  $x$  and  $y$  do not share a neighbor. If  $(x, y)$  is not an edge in  $G$  then set  $G := G - \bigcup_{i=1}^\ell V(T_i) + (x, y)$ ; otherwise, set  $G := G - \bigcup_{i=1}^\ell V(T_i)$ .

**Lemma 4.7** Let  $T_1, \dots, T_\ell$ ,  $\ell > 1$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . Suppose that a vertex in  $T_\ell$ , say  $w_\ell$ , does not share a common neighbor with  $v_1$  and does not share a common neighbor with  $w_1$ . Let  $w'_\ell$  be the nontriangle vertex that is a neighbor of  $w_\ell$ . Let  $G'$  be the graph resulting from  $G$  by removing the set of vertices  $\bigcup_{i=2}^\ell V(T_i)$  and adding the edge  $(w'_\ell, u_1)$ ; that is,  $G' = (G - \bigcup_{i=2}^\ell V(T_i)) + (w'_\ell, u_1)$ . Then  $\delta_{n(G)} = 3(\ell - 1)$ ,  $\delta_{\alpha(G)} \geq \ell - 1$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.

PROOF. The fact that  $\delta_{n(G)} = 3(\ell - 1)$  follows from the fact that  $|V(G)| = |V(G')| + |\bigcup_{i=2}^{\ell} V(T_i)|$ , and that  $\bigcup_{i=2}^{\ell} V(T_i)$  contains precisely  $3(\ell - 1)$  vertices.

To show that  $\delta_{\alpha(G)} \geq \ell - 1$ , let  $I'$  be a maximum independent set of  $G'$ . Since  $(w'_\ell, u_1) \in E(G')$ ,  $I'$  contains at most one vertex from  $\{w'_\ell, u_1\}$ . If  $I'$  excludes  $u_1$ , then  $I' \cup \{v_i : i = 2, \dots, \ell\}$  is an independent set in  $G$  of size  $|I'| + \ell - 1 = \alpha(G') + \ell - 1$ . On the other hand, if  $I'$  excludes  $w'_\ell$ , then  $I' \cup \{w_\ell\} \cup \{u_i : i = 2, \dots, \ell - 1\}$  is an independent set in  $G$  of size  $|I'| + \ell - 1 = \alpha(G') + \ell - 1$ . It follows that  $G$  has an independent set of size  $\alpha(G') + \ell - 1$ , and hence,  $\delta_{\alpha(G)} \geq \ell - 1$ .

Now since  $w'_\ell$  is not a neighbor of  $w_1$  nor of  $v_1$ , the addition of edge  $(w'_\ell, u_1)$  does not create a triangle. Since all vertices removed from  $G$  are triangle vertices, we have  $\delta_{nt(G)} \leq 0$ . By maximality of the path of triangles  $T_1, \dots, T_\ell$ , vertex  $w'_\ell$  is a nontriangle vertex, and  $T_1$  is not adjacent to any triangle in  $G'$ . Moreover, since  $T_1$  is a triangle in a path of at least two triangles,  $T_1$  does not share an edge with another triangle. It follows from the previous statements that edge  $(w'_\ell, u_1)$  cannot be an edge in an obstacle graph, and hence its addition does not create obstacle graphs. Now since at least one neighbor of each of  $w'_\ell$  and  $u_1$  was removed from  $G$ , we have  $\Delta(G') \leq 3$ .  $\square$

**Operation 4.6** Let  $T_1, \dots, T_\ell$ ,  $\ell > 1$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . Suppose that a vertex in  $T_\ell$ , say  $w_\ell$ , does not share a common neighbor with  $v_1$  and does not share a common neighbor with  $w_1$ . Let  $w'_\ell$  be the nontriangle vertex that is a neighbor of  $w_\ell$ . Then set  $G := (G - \bigcup_{i=2}^{\ell} V(T_i)) + (w'_\ell, u_1)$ .

**Lemma 4.8** Suppose that no two triangles in  $G$  are adjacent. Let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = d(v) = d(w) = 3$ . Let  $u'$ ,  $v'$ , and  $w'$  be the neighbors of  $u, v, w$ , respectively, that are not in the set  $\{u, v, w\}$ , and assume that no edge exists between any two vertices of  $\{u', v', w'\}$  (i.e., the subgraph of  $G$  induced by  $\{u', v', w'\}$  is an independent set). Suppose further that there are two vertices in  $\{u', v', w'\}$ , say  $u'$  and  $v'$ , that do not share a common neighbor in  $G$ . Let  $G'$  be the graph resulting from  $G$  by removing the set of vertices  $\{u, v, w\}$  and adding the edge  $(u', v')$ ; that is,  $G' = (G - \{u, v, w\}) + (u', v')$ . Then  $\delta_{n(G)} = 3$ ,  $\delta_{\alpha(G)} \geq 1$ , and  $\delta_{nt(G)} \leq 0$ . Moreover,  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.

PROOF. Note that since no two triangles in  $G$  are adjacent or share an edge, vertices  $u', v', w'$  are distinct nontriangle vertices.

It is clear that  $\delta_{n(G)} = 3$ . Since the two vertices in  $u', v'$  are adjacent in  $G'$ , any maximum independent set  $I'$  of  $G'$  contains at most one vertex from  $\{u', v'\}$ . If  $u' \notin I'$ , then  $I' \cup \{u\}$  is an independent set of  $G$ . On the other hand, if  $v' \notin I'$ , then  $I' \cup \{v\}$  is an independent set of  $G$ . It follows that  $\delta_{\alpha(G)} \geq 1$ .

Since  $u'$  and  $v'$  do not share a neighbor, the edge  $(u', v')$  is not a triangle edge in  $G'$ , and hence  $u'$  and  $v'$  are nontriangle vertices in  $G'$ . Since all the vertices removed from  $G$  are triangle vertices, we have  $\delta_{nt(G)} \leq 0$ . Now to show that  $G'$  is obstacle-free, it suffices to show that the addition of  $(u', v')$  does not create obstacle graphs. Since both  $u'$  and  $v'$  are nontriangle vertices in  $G'$ , the addition of  $(u', v')$  cannot create a small obstacle. If the addition of  $(u', v')$  creates a big obstacle, then this edge must be one of the two edges between two nontriangle vertices in the big obstacle. However, this would imply that there are two triangles in  $G$  that are adjacent; this contradicts the hypothesis of the lemma. Now since one neighbor of each of  $u'$  and  $v'$  was removed from  $G$ , we have  $\Delta(G') \leq 3$ .  $\square$

**Algorithm Simplify**

INPUT: A graph  $G$  with  $\Delta(G) \leq 3$  such that  $G$  is obstacle-free

OUTPUT: A graph  $G'$

1. **Repeat until** none of Operations 4.1– 4.7 applies to  $G$ :  
     pick the *first* operation in Operation 4.1, ..., Operation 4.7 in this order that applies to  $G$  and apply it;
2. **return** the resulting graph;

Figure 2: The algorithm **Simplify**.

**Operation 4.7** Suppose that no two triangles in  $G$  are adjacent, and let  $(u, v, w)$  be a triangle in  $G$  such that  $d(u) = d(v) = d(w) = 3$ . Let  $u'$ ,  $v'$ , and  $w'$  be the neighbors of  $u, v, w$ , respectively that are not in the set  $\{u, v, w\}$ . If there are two vertices in  $\{u', v', w'\}$ , say  $u'$  and  $v'$ , that do not share a common neighbor in  $G$ , then set  $G' := (G - \{u, v, w\}) + (u', v')$ .

**Proposition 4.9** Let  $G$  be a graph with  $\Delta(G) \leq 3$  such that  $G$  is obstacle-free. Let  $G'$  be the graph resulting from the application of the algorithm **Simplify** to  $G$ . Then the following are true:

- (i)  $\Delta(G') \leq 3$  and  $G'$  is obstacle-free.
- (ii) Every triangle vertex in  $G'$  has degree 3 (in  $G'$ ).
- (iii)  $\delta_{nt(G)} \leq 0$ , and hence  $nt(G') \geq nt(G)$ .
- (iv)  $\delta_{\alpha(G)} \geq \delta_{n(G)}/3$ .
- (v) No two triangles in  $G'$  share an edge or are adjacent.
- (vi) If  $(u, v, w)$  is a triangle in  $G'$  then each of  $u, v, w$  has exactly one neighbor  $u', v', w'$ , respectively, that is a nontriangle vertex. Moreover, vertices  $u', v', w'$  are distinct, no two of them are adjacent, and every two of them share a neighbor.

PROOF.

- (i) This follows from the fact that  $\Delta(G) \leq 3$  and from Lemma 4.2–Lemma 4.8, which state that the graph resulting from the application of each of Operations 4.1– 4.7 has maximum degree at most 3 and is obstacle-free.
- (ii) This follows from the fact that Operations 4.1 is not applicable to  $G'$ .
- (iii) This follows from Lemmas 4.2–4.8, which state that for each of Operations 4.1–4.7 we have  $\delta_{nt(G)} \leq 0$ .
- (iv) Each of Operations 4.1–4.7 removes  $3\ell$  vertices, for some integer  $\ell \geq 1$ , from the graph and guarantees that the size of the maximum independent set in the graph that the operation is applied to is at least larger by  $\ell$ , which is one third of the number of removed vertices, than the size of the maximum independent set of the graph resulting from the application of the operation. Therefore, if Operations 4.1–4.7 are applied to  $G$  to obtain  $G'$ , then  $\delta_{n(G)}$  vertices were removed from  $G$ , and  $\delta_{\alpha(G)} \geq \delta_{n(G)}/3$ .

- (v) Since  $G'$  is obstacle-free,  $G'$  does not contain a small obstacle as a subgraph, and hence, no two triangles in  $G'$  share an edge.

Since Operation 4.3 is not applicable to  $G'$ ,  $G'$  does not contain a cycle of triangles. Therefore, to show that no two triangles in  $G'$  are adjacent, it suffices to show that every maximal path of triangles in  $G'$  contains exactly one triangle.

Proceed by contradiction. Let  $T_1, \dots, T_\ell$ ,  $\ell > 1$ , be a maximal path of triangles, where  $T_i = (u_i, v_i, w_i)$  for  $i = 1, \dots, \ell$ , and  $u_i$  is adjacent to  $v_{i+1}$ , for  $i = 1, \dots, \ell - 1$ . By part (ii) of this proposition, all triangle vertices are of degree 3. Since Operation 4.6 is not applicable to  $G'$ , vertex  $w_\ell$  must share a neighbor  $x$  with one of the two vertices  $\{w_1, v_1\}$ , say  $w_1$ , and  $u_\ell$  must share a neighbor  $y$  with the other vertex  $v_1$ . Since Operation 4.5 is not applicable to  $G'$ ,  $x$  and  $y$  must share a neighbor  $z$ . Now since Operation 4.4 is not applicable to  $G'$ ,  $\ell \leq 2$ , and since  $\ell > 1$ , we have  $\ell = 2$ . But then the subgraph of  $G'$  induced by the set of vertices  $V(T_1) \cup V(T_2) \cup \{x, y, z\}$  is a big obstacle in  $G'$ , contradicting the fact that  $G'$  is obstacle-free (part (ii) of this proposition).

Therefore, any maximal path of triangles in  $G'$  contains exactly one triangle, and hence,  $G'$  does not contain adjacent triangles.

- (vi) Let  $(u, v, w)$  be a triangle in  $G'$ . Since every triangle vertex in  $G'$  is of degree 3 (part (ii) of this proposition) and no two triangles in  $G'$  are adjacent or share an edge (part (v) of this proposition), each of  $u, v, w$  has exactly one neighbor that is a nontriangle vertex; let these neighbors be  $u', v', w'$ , respectively, and note that since all these vertices are nontriangle vertices, they must be distinct. Since Operation 4.2 is not applicable to  $G'$ , no two vertices in  $u', v', w'$  are adjacent. Since no two triangles in  $G$  are adjacent or share an edge, and since Operation 4.7 is not applicable to  $G'$ , every two vertices in  $u', v', w'$  share a neighbor.

□

## 5 A combinatorial result

A block of a graph is called *difficult* [8] if it is isomorphic to one of the following four graphs (see Figure 3 for illustration):  $K_3$ ,  $C_5$ ,  $K_4$  with two of its edges each subdivided twice, or a graph arising from  $C_5$  by adding a new vertex and connecting it to three consecutive vertices of  $C_5$ . A connected graph is called *bad* [8] if every block of the graph is either a difficult block or an edge between two difficult blocks.

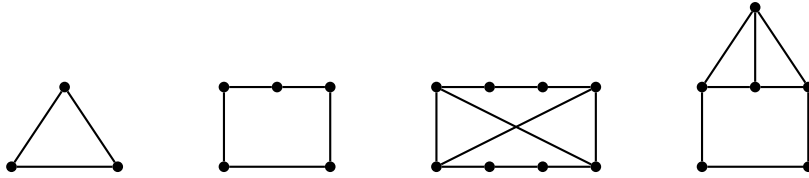


Figure 3: The difficult blocks.

Harant et al. [8] showed that if  $H$  is a  $K_4$ -free graph with  $\Delta(H) \leq 3$  then  $\alpha(H) \geq (4n(H) - e(H) - \lambda(H) - tr(H))/7$ , where  $\lambda(H)$  is the number of components of  $H$  that are bad, and  $tr(H)$

is the number of vertex-disjoint triangles in  $H$ . If  $H$  is connected, then  $H$  has 1 component, and if in addition  $H$  is not bad, then  $\lambda(H) = 0$ . Therefore, in this case the result of Harant et al. [8] implies the following:

**Lemma 5.1** ([8]) *If  $H$  is a  $K_4$ -free connected graph with maximum degree at most 3 such that  $H$  is not bad, and if  $G$  has at most  $\text{tr}(H)$  vertex-disjoint triangles then  $\alpha(H) \geq (4n(H) - e(H) - \text{tr}(H))/7$ .*

**Theorem 5.2** *Let  $G$  be an obstacle-free graph with  $\Delta(G) \leq 3$ . Then  $\alpha(G) \geq n(G)/3 + nt(G)/168$ .*

PROOF. Let  $G'$  be the graph resulting from applying the algorithm **Simplify** to  $G$ . By part (iv) of Proposition 4.9, we have  $\delta_{\alpha(G)} \geq \delta_{n(G)}/3$ . By part (iii) of Proposition 4.9, we have  $\delta_{nt(G)} \leq 0$ . Therefore, to prove the theorem, it suffices to show that  $\alpha(G') \geq n(G')/3 + nt(G')/168$ .

Let  $T'$  be the subgraph of  $G'$  consisting of the vertices and edges that appear in the triangles of  $G'$ . By part (v) of Proposition 4.9, no two triangles in  $G'$  share an edge or are adjacent, and hence  $T'$  consists of disjoint triangles. Let  $R'$  be the subgraph of  $G'$  induced by the set of nontriangle vertices in  $G'$  (i.e., the vertices in  $V(G') - V(T')$ ). Then  $R'$  is a triangle-free graph. By part (vi) of Proposition 4.9, every vertex in  $T'$  is of degree 3 (in  $G'$ ), and has exactly one neighbor in  $R'$ ; for a vertex  $u \in T'$ , we denote its neighbor in  $R'$  by  $u'$ . Note that for two distinct vertices  $u, v$  in  $V(T')$  that are not vertices of the same triangle,  $u'$  can be equal to  $v'$ .

Let  $(u, v, w)$  be a triangle in  $T'$ . Consider the vertices  $u', v', w'$  and note that by part (vi) of Proposition 4.9, these vertices are distinct, no two of them are adjacent, and every two of them share a common neighbor in  $R'$ . Note that the common neighbor must be in  $R'$ , and that the three vertices  $u', v', w'$  could share the same common neighbor.

Let  $u' \in V(R')$  be a vertex that is adjacent to some triangle vertex in  $T'$ . We claim that  $u'$  has exactly one neighbor in  $T'$ , unless the graph  $G'$  has a component of exactly 10 vertices and an independent set of size 4. In effect, let  $(u, v, w)$  and  $(p, q, r)$  be two distinct triangles in  $T'_2$  such that  $u'$  is a neighbor of both  $u$  and  $p$ . Then  $u'$  must share a common neighbor with each of  $v', w', q', r'$ . Since  $u, v, w$  are distinct vertices, and  $p, q, r$  are distinct vertices, and since the degree of  $u'$  is at most 3, this is only possible if  $q' = v'$  and  $r' = w'$  (resp.  $q' = w'$  and  $r' = v'$ ) and there exists a vertex  $x$  in  $R'$  that is adjacent to  $u', v', w'$ . In this case the degree of each of the vertices  $u, v, w, p, q, r, u', v', w', x$  in  $G'$  must be 3, and hence the subgraph  $C$  of  $G'$  induced by these vertices must be a connected component of  $G'$ . It is easy to see that the set of vertices  $\{p, u, v', w'\}$  is an independent set in  $C$  of size 4. Since  $n(C) = 10$ ,  $nt(C) = 4$ , and  $\alpha(C) = 4$ , it follows in this case that  $\alpha(C) \geq n(C)/3 + nt(C)/168$ . We call such components in  $G'$  *special components*; see Figure 4 for illustration.

Now let  $(u, v, w)$  be a triangle in  $T'$  that is not contained in a special component. From the above discussion, each of  $u', v', w'$  has exactly one neighbor in  $G'$ , namely vertices  $u, v, w$ , respectively. We also know that the vertices  $u', v', w'$  are distinct, and every two of them share a neighbor in  $R'$ . Again note that these three vertices can share the same common neighbor in  $R'$ . We associate with triangle  $(u, v, w)$  the set of vertices in  $R'$  consisting of  $u', v', w'$ , plus each vertex in  $R'$  that is a common neighbor of the two vertices in one of the pairs  $\{(u', v'), (u', w'), (v', w')\}$ ; denote this set of vertices by  $S_{uvw}$ , and note that  $|S_{uvw}| \geq 4$ , and that the subgraph of  $G'$  induced by  $u, v, w$  plus the vertices in  $S_{uvw}$  is 2-connected. More importantly, for two distinct triangles  $(u, v, w)$  and  $(p, q, r)$  in  $T'$ , their associated sets  $S_{uvw}$  and  $S_{pqr}$  are disjoint. This is true because the two sets  $\{u', v', w'\}$  and  $\{p', q', r'\}$  are disjoint, and because every vertex in  $G'$  has degree at most 3, and hence no vertex in  $R'$  can be a common neighbor of two vertices in  $\{u', v', w'\}$  and two vertices in

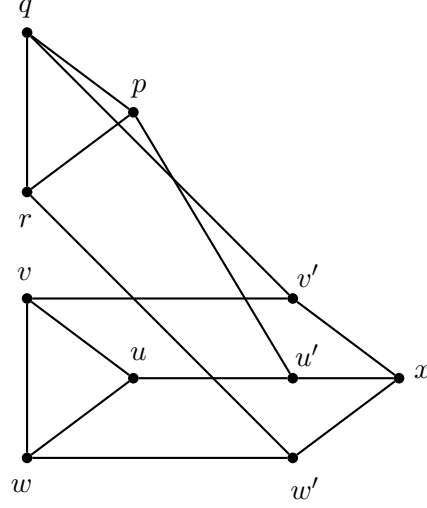


Figure 4: A *special component*.

$\{p', q', r'\}$  at the same time. Therefore, for every triangle  $(u, v, w) \in T'$  we can correspond to it, in a one-to-one fashion, the set  $S_{uvw}$ .

Let  $C$  be a connected component in  $G'$  that contains (at least) a triangle  $(u, v, w) \in T'$ . If  $C$  is a special component then  $\alpha(C) \geq n(C)/3 + nt(C)/168$  as explained above. If  $C$  is not a special component, then since the subgraph induced by  $u, v, w$  plus the set of vertices in  $S_{uvw}$  is 2-connected, this set is a subgraph of some block in  $C$ , which clearly cannot be isomorphic to one of the graphs in Figure 3. Therefore, the component  $C$  is not bad, and by Lemma 5.1, we have  $\alpha(C) \geq (4n(C) - e(C) - tr(C))/7$ . (Note that the preconditions of the lemma are satisfied since  $C$  is connected, and does not contain a small obstacle, and hence is  $K_4$ -free.) Since  $C$  has maximum degree at most 3, we have  $e(C) \leq 3n(C)/2$ . Since every triangle  $(u, v, w)$  in  $C$  can be corresponded with the set  $S_{uvw}$  of cardinality at least 4, in a one-to-one fashion, such that for any two distinct triangles their corresponding sets are disjoint,  $tr(C) \leq n(C)/7$ , and hence  $n(C) \leq 7nt(C)/4$ . Combining all the above we obtain:

$$\begin{aligned}
\alpha(C) &\geq (4n(C) - e(C) - tr(C))/7 \\
&\geq (4n(C) - 3n(C)/2 - (n(C) - nt(C))/3)/7 \\
&\geq 5n(C)/14 - (n(C) - nt(C))/21 \\
&\geq n(C)/3 + nt(C)/21 - n(C)/42 \\
&\geq n(C)/3 + nt(C)/168.
\end{aligned}$$

Now for any component  $C$  that does not contain any triangle from  $T'$ ,  $C$  is a triangle-free

graph. It follows from [9] that  $\alpha(C) \geq 5n(C)/14 = n(C)/3 + n(C)/42 \geq n(C)/3 + nt(C)/42 \geq n(C)/3 + nt(C)/168$  in this case.

By summing over all components in  $G'$ , we obtain  $\alpha(G') \geq n(G')/3 + nt(G')/168$ . This completes the proof.  $\square$

The result in Theorem 5.2 can be further improved:

**Theorem 5.3** *Let  $G$  be an obstacle-free graph with  $\Delta(G) \leq 3$ . Then  $\alpha(G) \geq n(G)/3 + nt(G)/63$ .*

**PROOF.** If none of Operations 4.1–4.7 applies to the graph, then every triangle in the graph must be contained in one of the two subgraphs depicted in Figure 5; we call the graph on the left a *type-I steeple* and the one on the right a *type-II steeple*. We can apply more operations to simplify the graph further. Those operations are depicted in Figure 6. Each of these operations removes a subgraph  $H$  from  $G$  (the subgraph induced by the set of solid vertices plus the set of shaded vertices in the figures) to obtain a subgraph  $G'$  (i.e.,  $G' = G - V(H)$ ) such that there exists a subset of vertices  $S_H \subseteq V(H)$  that is an independent set (the set of solid vertices) satisfying: (1)  $\alpha(G) \geq |S_H| + \alpha(G')$ , (2)  $|S_H| \geq n(H)/3 + nt(H)/63$ , and (3)  $nt(G) = nt(H) + nt(G')$ . We then append the operations in Figure 6 to Operations 4.1–4.7 in the algorithm **Simplify**. The resulting graph  $G'$  after the application of the algorithm satisfies the following conditions, which allow us to obtain a better upper bound on the number of triangles in  $G'$ —and hence a better lower bound on  $nt(G')$ . If for a steeple  $S$  we denote by  $N(S)$  the neighbors of the vertices of  $V(S)$  that are in  $G - V(S)$ , then for any steeple  $S$  in the resulting graph  $G'$ :  $|N(S)| = 3$ , every vertex in  $N(S)$  is of degree 3, and  $N(S)$  is an independent set; and for any two steeples  $S_1$  and  $S_2$  in  $G'$ :  $V(S_1) \cup N(S_1)$  is disjoint from  $V(S_2) \cup N(S_2)$ , and no edge exists between a vertex in  $N(S_1)$  and a vertex in  $N(S_2)$ . The above properties allow us to correspond, in a one-to-one fashion, with every triangle in  $G'$  9 vertices in  $G'$ . Using a similar analysis to the one used in the proof of Theorem 5.2, we can show that the resulting graph  $G'$  after the application of the algorithm satisfies  $tr(G') \leq n(G')/12$ , and that  $\alpha(G') \geq n(G')/3 + nt(G')/63$ . This, together with properties (1)–(3) mentioned above, give  $\alpha(G) \geq n(G)/3 + nt(G)/63$ .  $\square$

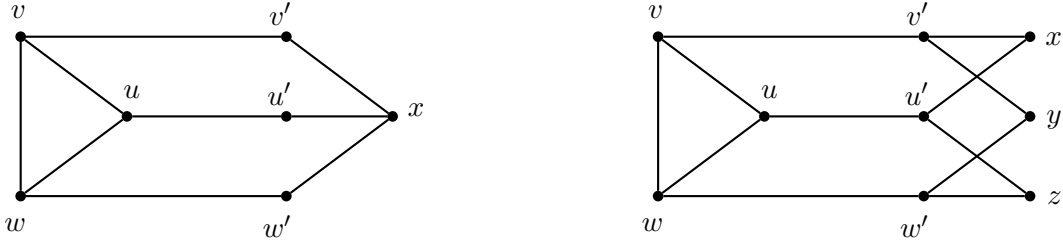


Figure 5: The *steeple graphs*. The graph on the left is referred to as a *type-I steeple* and that on the right as a *type-II steeple*. Note that no edges exist between any two vertices in  $\{u', v', w'\}$  in both type-I and type-II steeples. Note also that the vertices  $u', v', w'$  in a type-I steeple could be either of degree 2 or 3; similarly, the vertices  $x, y, z$  in a type-2 steeple could be either of degree 2 or 3.

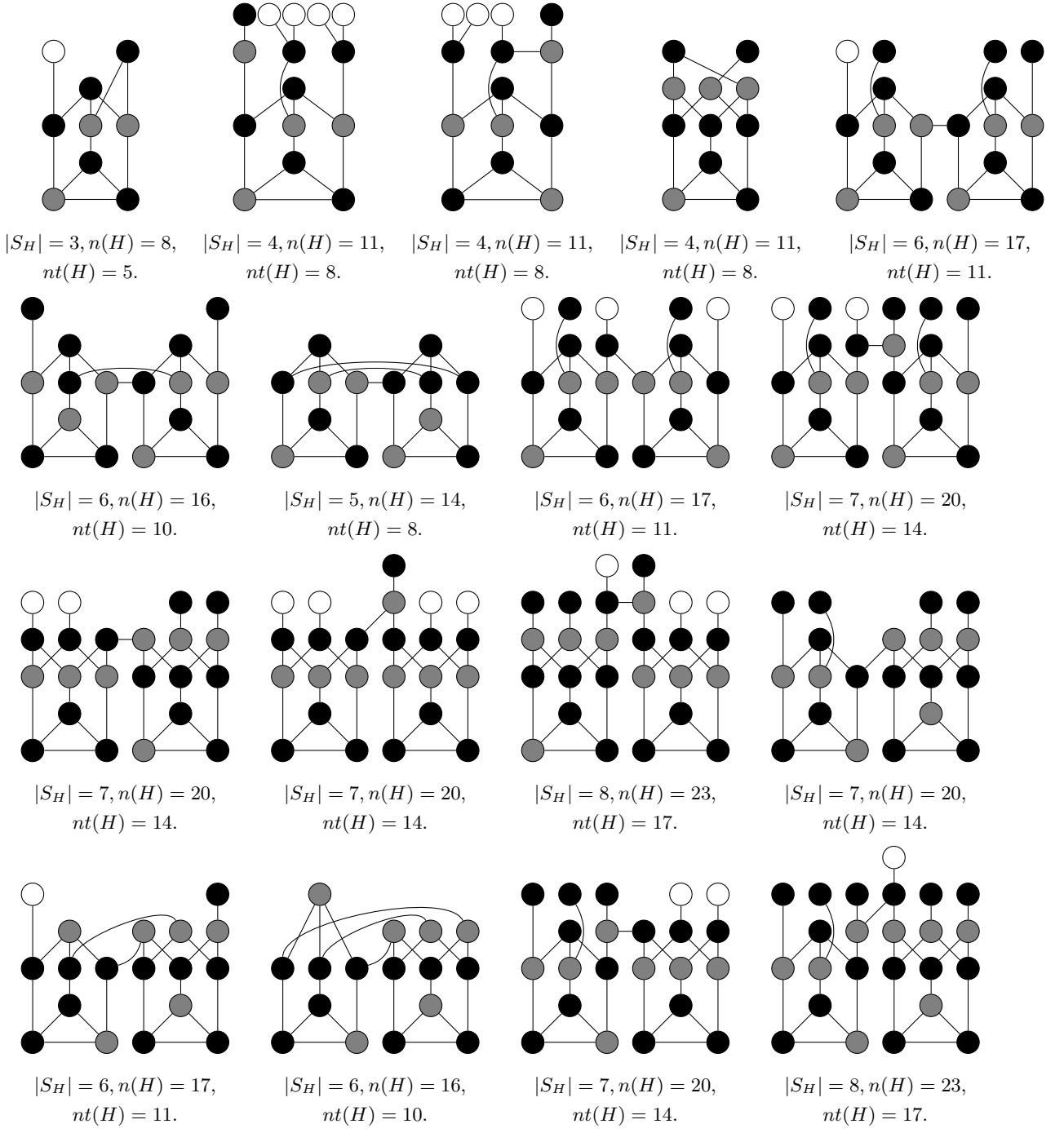


Figure 6: The appended operations.



**Algorithm Kernelize**INPUT: An instance  $(G, k)$  of IS-3OUTPUT: An instance  $(G', k')$  of IS-3

1. **if**  $k \leq n(G)/4$  **then accept** the instance  $(G, k)$ ;
2. **apply** Reduction Rules 3.1–3.4 to the instance  $(G, k)$  until none of them is applicable;
3. let  $(G', k')$  be the resulting instance;
4. **if**  $k' \leq 211n(G')/630$  **then accept** the instance  $(G, k)$ ;  
**else return** the instance  $(G', k')$ .

Figure 7: The algorithm **Kernelize**.

## 6 The kernel

Consider the algorithm given in Figure 7.

**Theorem 6.1** *Given an instance  $(G, k)$  of IS-3, the algorithm **Kernelize** either accepts the instance  $(G, k)$  correctly or returns an equivalent instance  $(G', k')$  of IS-3 such that  $n(G') \leq 630k'/211$ . The running time of the algorithm **Kernelize** is  $O(k)$ .*

PROOF. Since  $\Delta(G) \leq 3$ ,  $G$  is 4-colorable and  $\alpha(G) \geq n(G)/4$ . Therefore, if  $k \leq n(G)/4$  then the algorithm **Kernelize** can accept the instance  $(G, k)$  directly. It follows that step 1 of the algorithm is correct.

Let  $(G', k')$  be the instance of IS-3 resulting from  $(G, k)$  after step 2 of the algorithm **Kernelize**. The validity of Reduction Rules 3.1–3.4 follows from Facts 3.1–3.2 and Lemmas 3.3–3.4, given in Section 3. Therefore,  $(G', k')$  is an instance of IS-3 that is equivalent to the instance  $(G, k)$ . Since Reduction Rule 3.2 does not apply to  $G'$ ,  $G'$  does not contain small obstacles (see Figure 1). Since Reduction Rule 3.4 does not apply to  $G'$ ,  $G'$  does not contain big obstacles. It follows that  $G'$  is obstacle-free. By Theorem 5.3,  $\alpha(G') \geq n(G')/3 + nt(G')/63$ . Since  $G'$  is a reduced graph (see Definition 3.1), by Lemma 3.6 we have  $nt(G') \geq n(G')/10$ . It follows from the previous two statements that  $\alpha(G') \geq 211n(G')/630$ , or equivalently,  $n(G') \leq 630\alpha(G')/211$ . Therefore, if  $k' \leq 211n(G')/630$ , then  $G'$  has an independent set of size  $k'$ , and equivalently  $G$  has an independent set of size  $k$ ; therefore the algorithm **Kernelize** can accept the instance  $(G, k)$ . If this is not the case, then the algorithm returns the instance  $(G', k')$  in which  $n(G') < 630k'/211$ .

To argue that the running time of the algorithm is  $O(k)$ , note that after step 1 of the algorithm we have  $k > n(G)/4$ , or equivalently,  $n(G) < 4k$ . Now that the size of the graph is  $O(k)$ , it is not difficult to see that step 2 of the algorithm can be implemented to run in  $O(k)$  time with the help of some suitable data structures. As a matter of fact, it is not difficult to see that Reduction Rules 3.1 and 3.2 can be implemented to run in  $O(k)$  time overall (throughout the whole execution of step 2). Moreover, with the help of an auxiliary graph whose vertices correspond to the triangles of  $G$  and whose edges correspond to adjacent triangles in  $G$ , which can be created and maintained in  $O(k)$  time, Reduction Rules 3.3 and 3.4 can also be implemented to run in  $O(k)$  time overall.

This completes the proof.  $\square$

**Corollary 6.2** *The IS-3 problem has a kernel of size at most  $630k/211 < 3k$  that is computable in  $O(k)$  time.*

## 7 Kernel lower bounds

A *vertex cover* in a graph  $G$  is a set of vertices in  $V(G)$  such that each edge in  $E(G)$  is incident on at least one vertex in this set. The VERTEX COVER problem on graphs of maximum degree at most 3, abbreviated VC-3, is defined as follows:

VC-3. Given an undirected graph  $G$  with  $\Delta(G) \leq 3$ , and a nonnegative integer  $k$ , determine if  $G$  has a vertex cover of size at most  $k$ .

The upper bound results on the kernel size for IS-3 in Theorem 6.1 give a lower bound on the kernel size for VC-3.

Let  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  be two dual parameterized problems.<sup>1</sup> The following result was shown in [4]:

**Lemma 7.1** ([4]) *If  $\mathcal{Q}_1$  has a kernel of size  $c_1k$  and  $\mathcal{Q}_2$  has a kernel of size  $c_2k$ , then unless  $P=NP$ ,  $c_1$  and  $c_2$  must satisfy  $(c_1 - 1)(c_2 - 1) \geq 1$ .*

Moreover, it was shown in [4] that the INDEPENDENT SET and the VERTEX COVER problem are dual problems. It follows that the restrictions of INDEPENDENT SET and VERTEX COVER to graphs of maximum degree at most 3 are dual problems. Based on the previous statement, Lemma 7.1, and on Theorem 6.1, we derive the following result:

**Theorem 7.2** *Unless  $P=NP$ , the VC-3 problem does not have a kernel of size at most  $630k/419$ .*

## References

- [1] H. Bodlaender, R. Downey, M. Fellows, and D. Hermelin. On problems without polynomial kernels. *J. Comput. Syst. Sci.*, 75(8):423–434, 2009.
- [2] Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (Meta) kernelization. In *FOCS*, pages 629–638. IEEE Computer Society, 2009.
- [3] R. Brooks. On colouring the nodes of a network. *Math. Phys. Sci.*, 37(4):194197, 1941.
- [4] J. Chen, H. Fernau, I. Kanj, and G. Xia. Parametric duality and kernelization: Lower bounds and upper bounds on kernel size. *SIAM Journal on Computing*, 37(4):1077–1106, 2007.
- [5] R. Downey and M. Fellows. *Parameterized Complexity*. Springer, New York, 1999.
- [6] S. Fajtlowicz. On the size of independent sets in graphs. *Congr. Numer.*, 21:269274, 1978.
- [7] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete problems. In *Proceedings of the sixth annual ACM symposium on Theory of computing*, pages 47–63. ACM, 1974.
- [8] J. Harant, M. Henning, D. Rautenbach, and I. Schiermeyer. The independence number in graphs of maximum degree three. *Discrete Mathematics*, 308(23):5829–5833, 2008.
- [9] C. Heckman and R. Thomas. A new proof of the independence ratio of triangle-free cubic graphs. *Discrete Mathematics*, 233(1-3):233–237, 2001.

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<sup>1</sup>The formal definition of a dual problem can be found in [4].

- [10] K. Jones. Size and independence in triangle-free graphs with maximum degree three. *Journal of Graph Theory*, 14(5):525–535, 1990.
- [11] W. Staton. Some Ramsey-type numbers and the independence ratio. *Transactions of the American Mathematical Society*, 256:353–370, 1979.
- [12] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.